

# The general QCD parametrization and the $1/N_c$ expansion: A comparison

G.Dillon and G.Morpurgo

Università di Genova and Istituto Nazionale di Fisica Nucleare, Genova, Italy.

**Abstract.** A comparison is presented of the two methods mentioned in the title for treating hadron properties in QCD. While the general parametrization is derived exactly from real QCD, the equivalence of the large  $N_c$  description to real QCD, with 3 colors, is questionable. The reason why in some cases the large  $N_c$  method approximately works (while in others does not) is clarified.  
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## 1. Introduction.

A paper by Buchmann and Lebed [1] compares, in a specific case, the  $1/N_c$  description and the general parametrization (GP) method of QCD [2, 3]. The GP method is presented in [1], for the case at hand, as a very good (but not exact) approximation to the  $1/N_c$  method, which is regarded as fundamental. This conclusion is seriously misleading, but Ref.[1] is useful because, on comparing the two methods, one can clarify the problems that affect the large  $N_c$  description.

Both the  $1/N_c$  and the GP methods are spin-flavor parametrizations of hadronic properties; but the GP method is derived directly from real (3 colors) QCD, while no derivation from real QCD exists for the large  $N_c$  method. There, all is based on assuming that a result of 't Hooft, valid for QCD in the  $N_c = \infty$  limit in 1+1 dimensions, supports a way of dealing with 4-dimensional 3 colors QCD. One (of the many) reason(s) for doubting the  $1/N_c$  method was expressed in [4]: “The basis for the large  $N_c$  approach is the assumption that  $N_c = 3$  QCD is similar to QCD in

the limit  $N_c = \infty$ . In particular it is assumed that there are no phase transitions as we go from  $N_c = 3$  to  $N_c \rightarrow \infty$ . Currently the status of these assumptions is not clear, because not much is known about  $\text{QCD}(N_c = \infty)$ . Here we will show on an example that the  $1/N_c$  method may indeed lead to incorrect results.

In what follows we shall first consider the  $1/N_c$  method in two cases where it works, next in a case where it has some problems and finally consider an example (Sect.4) where it fails. But, before, we note two points:

1. The GP method is an exact consequence of QCD, based only on few general properties of the QCD Lagrangian. For many physical quantities (e.g. masses, magnetic moments, electromagnetic and semileptonic matrix elements, e.m. form factors etc.) of the lowest multiplets of hadrons -excited states will not be considered here- it leads to an exact spin-flavor parametrization. The GP method was developed [2,3] precisely to explain the unexpected semiquantitative success of the non relativistic quark model (NRQM) [5]; it accomplished this [2] long before the use of the  $1/N_c$  method for the same problem. It emerged that the structure of the terms in the GP is similar to that of the NRQM. Because GP terms of increasing complexity have decreasing coefficients, often few terms reproduce the data fairly well, showing why then the NRQM works already in its most naive form.

2. Although  $SU_6$  was important in leading to the NRQM [5], it does not play a role after that. In constructing [5] the NRQM it was essential that the space part of the octet and decuplet baryon wave functions has an overall zero orbital angular momentum:  $L = 0$  and that the baryon **8+10** NRQM states factorize:

$$\phi_B = X_{L=0}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \cdot W_B(s, f) \quad (1)$$

where  $X$  is the space part and  $W_B(s, f)$  are the spin-flavor factors. (Color is understood.) The  $W_B$ 's are symmetric in the three quark variables and, because  $L = 0$ , have necessarily  $J = 1/2$  and  $J = 3/2$  for the octet and decuplet, so that, *automatically the factorization implies* that the  $W_B$  factor of  $\phi_B$  has the  $SU_6$  form,

without the need of invoking at all  $SU_6$ . As it will appear, the factorizability of  $\phi_B$  (1) is essential to derive the simple structure of the GP. In the GP there is no need to relate the states to  $SU_6$  representations, as in the  $1/N_c$  method; nor to introduce, as in [1], a new type of quarks, the “representation quarks”.

We recall some notation of the GP method [2,3]:  $|\phi_B\rangle$  is, in the quark-gluon Fock space, the state of three quarks with wave function  $\phi_B$  and no gluons. The exact eigenstate of the QCD hamiltonian  $H_{QCD}$  for baryon  $B$  (with mass  $M_B$ ) at rest is written  $|\psi_B\rangle$ . It is  $H_{QCD}|\psi_B\rangle = M_B|\psi_B\rangle$ . A unitary transformation  $V$  (see [2]) acting on  $|\phi_B\rangle$ , transforms it into the exact state  $|\psi_B\rangle$  of  $H_B$ , so that:

$$|\psi_B\rangle = |qqq\rangle + |qqq\bar{q}q\rangle + |qqq, Gluons\rangle + \dots \quad (2)$$

where the last form of (2) recalls that  $V|\phi_B\rangle$  is a superposition of all possible quark-antiquark-gluon states with the correct quantum numbers. In particular, configuration mixing is automatically included in  $V|\phi_B\rangle$ . The mass of a baryon is:

$$\begin{aligned} M_B &= \langle\psi_B|H_{QCD}|\psi_B\rangle = \langle\phi_B|V^\dagger H_{QCD}V|\phi_B\rangle = \\ &= \langle W_B | \text{“parametrized mass”} | W_B \rangle \end{aligned} \quad (3)$$

The last step (eliminating the space variables) is due to the factorizability of  $\phi_B$  (eq.(1)). In the next section we discuss the *parametrized mass* in (3).

## 2. The hierarchy of the parameters in the GP method.

Although to explain why the NRQM works was the aim and original achievement of the GP, the method led to other results, on exploiting [2,3] a fact that emerged from the data, the hierarchy of the parameters. In (3) the “*parametrized mass*” of the **8+10** baryons derived with the GP is [2,3] (in the notation of [3]):

$$\text{“parametrized mass”} = M_0 + B \sum_i P_i^s + C \sum_{i>k} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) +$$

$$\begin{aligned}
& + D \sum_{i>k} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) (P_i^s + P_k^s) + E \sum_{\substack{i \neq k \neq j \\ (i > k)}} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) P_j^s + a \sum_{i>k} P_i^s P_k^s + \\
& + b \sum_{i>k} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) P_i^s P_k^s + c \sum_{\substack{i \neq k \neq j \\ (i > k)}} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) (P_i^s + P_k^s) P_j^s + d P_1^s P_2^s P_3^s
\end{aligned} \tag{4}$$

In (4)  $P_i^s$ 's are projectors on the strange quarks;  $M_0, B, C, \dots, d$  are parameters. Only  $(a+b)$  intervenes in the masses. The  $u-d$  mass difference is neglected.

A comment on (4): Because the different masses of the lowest octet and decuplet baryons are 8 (barring e.m. and isospin corrections), Eq.(4), with 8 parameters ( $M_0, B, C, D, E, a+b, c, d$ ), is certainly true, no matter what is the underlying theory. Yet the general parametrization (4) is not trivial: The values of the above 8 parameters are seen to decrease strongly on moving to terms with increasing number of indices (Eq.(5)). In deriving (4) from QCD, the term  $\Delta m \bar{\psi} P^s \psi$  in the QCD Lagrangian is treated exactly; Eq. (4) is correct to all orders in flavor breaking and in it all possible diagrams, including closed loops, are taken into account. In (4) the parameters (in MeV) (obtained from the pole masses) are:

$$\begin{aligned}
M_0 &= 1076 & , & \quad B = 192 & , & \quad C = 45.6 & , & \quad D = -13.8 \pm 0.3 \\
(a+b) &= -16 \pm 1.4 & , & \quad E = 5.1 \pm 0.3 & , & \quad c = -1.1 \pm 0.7 & , & \quad d = 4 \pm 3
\end{aligned} \tag{5}$$

The hierarchy of these numbers is evident; it corresponds [3]<sup>1</sup> to a reduction factor 0.37 for an additional pair of indices (additional exchanged gluon- ref.6f of [3], fig.1) and 0.3 for each flavor breaking factor  $P_i^s$ . Neglecting in (4)  $c$  and  $d$ , the following formula results (ref.6d of [3]), a generalization of the Gell-Mann Okubo formula that includes octet and decuplet:

$$\frac{1}{2}(p + \Xi^0) + T = \frac{1}{4}(3\Lambda + 2\Sigma^+ - \Sigma^0) \tag{6}$$

Symbols stay for masses and  $T$  is the following combination of decuplet masses:

$T = \Xi^{*-} - (\Omega + \Sigma^{*-})/2$ . Because of the level of accuracy reached, we wrote (6) so

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<sup>1</sup>An additional pair of indices in a mass term of the GP implies the exchange of at least an additional gluon, producing a reduction factor from 0.21 to 0.37 estimated fitting the Eq.(4) with parameters obtained respectively using the conventional and pole masses; we prefer ([3],ref.12) the latter determination.

as to be free of electromagnetic effects (the combinations in (6) are independent of electromagnetic and isospin effects, to zero order in flavor breaking.) The data satisfy (6) as follows:  $l.h.s. = 1133.1 \pm 1.0$  ;  $r.h.s. = 1133.3 \pm 0.04$ .

A hierarchy analogous to that for the masses (related to the number of indices, that is of gluons exchanged and of the  $P^s$ ), results in the GP of several other quantities [2,3,6]; for instance the magnetic moments, the  $\gamma$  decays of vector mesons, the semileptonic decays, the baryon e.m. form factors, etc. Thus the GP allows to analyze several hadronic properties. However (think in terms of Feynman diagrams) there is no reason at all to expect the reduction factor for the exchange of an additional gluon (call it  $R_g$ ) to be in all cases precisely equal to that for the baryon masses, 0.37; for instance for the baryon magnetic moments  $R_g$  is of order  $0.2 \pm 0.02$ . As to the flavor reduction factor ( $R_f$  in the following), this is essentially the same (0.3 to 0.33) in all cases. <sup>2</sup>

A final remark (see [3]) clarifies the meaning of the coefficients in eq.(4): A QCD calculation would express each ( $M_0$ ,  $B \dots c$ ,  $d$ ) in (4) in terms of the quantities in the QCD Lagrangian, e.g. the running quark masses -normalized at (say)  $q \approx 1$  GeV- and the dimensional (mass) parameter  $\Lambda \equiv \Lambda_{QCD}$ ; for instance, setting for simplicity  $m_u = m_d = m$ , one has:  $M_0 \equiv \Lambda \hat{M}_0(m/\Lambda, m_s/\Lambda)$  where  $\hat{M}_0$  is some function. Similarly for  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $a$ ,  $b$ ,  $c$ ,  $d$ . The numerical values of the coefficients should be seen as the result of a QCD exact calculation, performed with an arbitrary choice of the renormalization point of the running quark masses.

### 3. A comparison with the large $N_c$ method.

We now compare the parametrized baryon mass (4), with the same quantity obtained in the  $1/N_c$  method. There (ref.[7], Eq.3.4) the parametrization of the

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<sup>2</sup>The above value of  $R_g$  is obtained from the average of  $|g_6|$ ,  $|g_5|$ ,  $|g_4|$ , which are all reduced with respect to  $|g_1|$  by a factor  $R_g \cdot R_f$ .

baryon masses is also expressed in terms of 8 parameters (from  $c_{(0)}^{1,0}$  to  $c_{(3)}^{64,0}$ ), but these parameters multiply collective rather than individual quark variables. Again, setting to zero the smaller coefficients, one finds a relation between octet and decuplet baryon masses (namely Eq.(4.6) in [7]). It was not noted, in [7] (nor in [8])-see also [9]- that such Eq.(4.6) in [7] coincides -except for the notation and the use of the Okubo second order relation- with the Eq.(6) above.

Thus the  $1/N_c$  method is characterized by a hierarchy, for the masses, similar to that of the GP; but note that while the large  $N_c$  description fixes the hierarchy at  $1/N_c$ , in the GP method  $1/3$  is just an *order of magnitude*, for the reduction factor of the baryon masses. As to the closed loops, corresponding in the GP to Trace terms (see ref.14 in [3]), their contribution is negligible or not, depending on the number of gluons that enter in the loop, due to the Furry theorem (see [10], in particular fig.1). Incidentally, the Trace terms in Ref.[6] must be there; they can be neglected only because they are depressed due to the Furry theorem implying the exchange of many gluons. The statement on this in [1] (after Eq.(3.6)) is confusing, as it is misleading the assertion, there, after Eq.(3.5), that by neglecting, as we did in [6], terms proportional to  $m_u - m_d$  (which [3] are of order  $|m_u - m_d|/(\xi\Lambda_{QCD}) \approx 5 \cdot 10^{-3}$ ) we imposed a “mild physical constraint”. Anyhow the conclusion of Ref.[1] seems to be that the relation between the radii of  $n, p$  and  $\Delta$  implied by the GP and  $1/N_c$  methods is the same.

The question is now: Does the large  $N_c$  method always lead to the same results as the GP method? The answer is negative, and we shall illustrate below why in some cases the  $1/N_c$  has little basis or does not work.

First we comment on the Coleman-Glashow (CG) relation considered in ref.[11]. In ref.[11], we showed that neither the  $u - d$  mass difference, nor the Trace terms, modify the conclusion, reached in ref.6e of [3], that only comparatively few three index flavor terms violate the CG relation. This explains the “miraculous” preci-

sion of the CG relation, originally derived in exact  $SU(3)_f$ ; a precision confirmed by a recent measurement of the  $\Xi^0$  mass [12].

After the appearance of [11](as hep-ph/004198), a preprint by Jenkins and Lebed [13] implied that in the large  $N_c$  description it is “natural” (not “miraculous”) that the CG relation is so beautifully verified. It is asserted in [13] that the terms neglected are “naturally” expected to be small.

This confidence, however, has little basis. For the CG relation the terms in the GP are many [11]. It is unjustified to estimate their global contribution only by the order in  $1/N_c$  of a typical term, as done in the  $1/N_c$  method. Due also to this, the predictions of the  $1/N_c$  expansion do not have a real QCD foundation.

To summarize: In two of the three cases considered so far ( $N$ ,  $\Delta$  charge radii [6,1] and the octet-decuplet mass formula) the reason why the  $N_c$  description reproduces the GP results is that the number of parameters in the GP equals that in the  $1/N_c$  treatment. For the third case, the CG relation, for each order in  $1/N_c$ , many terms contribute, and we reiterate, therefore, what stated above.

A simple case where the results of the large  $N_c$  method clearly differ from the GP analysis is that of the magnetic moments of p,n and  $\Delta$  's. There the  $1/N_c$  method cannot account for the facts, contrary to the assertions in refs.[14-16]. Because it omits effects of order  $1/N_c^2$ , the  $1/N_c$  expansion is unable to account, for instance, for the  $\mu(p)/\mu(n)$  ratio and for the  $\Delta \rightarrow p\gamma$  transition.

#### 4. The magnetic moments.

The most general QCD expression for the spin-flavor structure of the magnetic moments of the **8** and **10** non strange baryons B is [2,17]:

$$\mu(B) = \langle W_B | \sum_{perm} [\alpha Q_1 + \delta(Q_2 + Q_3)] \sigma_{1z} + [\beta Q_1 + \gamma(Q_2 + Q_3)] \sigma_{1z} (\sigma_2 \cdot \sigma_3) | W_B \rangle \quad (7)$$

The Eq.(7) is the same as Eq.(62) of [2];  $\alpha, \delta, \beta, \gamma$  are four real parameters. We left

out in (7) the Trace terms [3],negligible in the present situation, as well as effects of order  $(m_d - m_u)$ . The sum over perm(utations) in (7) means that to the term (123) displayed one adds (321) and (231).<sup>3</sup> As shown in Sect.2 we have:

$$|\alpha| = (5 \pm 0.5)|\delta| \quad (8)$$

and, as we shall extract below from the data:  $|\delta| \approx 3|\beta| \approx 3|\gamma|$ .

The presence of the four parameters in QCD contrasts with the  $1/N_c$  treatment, where only two or three parameters (including the Trace term) appear [14-16]; we come back to this below. First we examine some consequences of the above four parameters. Using particle symbols for the magnetic moments (in proton magnetons) consider  $p, n, \Delta$  extracted from (7) and the  $(\Delta \rightarrow p\gamma)_0$  matrix element extrapolated to the vanishing transferred photon momentum ( $k = 0$ ).We obtain:

$$p = (\alpha - 3\beta - 2\gamma) , \quad n = -(2/3)(\alpha - \delta - 2\beta + 2\gamma) , \quad \Delta = (\alpha + 2\delta + \beta + 2\gamma)Q_\Delta \quad (9)$$

$$(\Delta \rightarrow p\gamma)_0 = (2/3)\sqrt{2}(\alpha - \delta + \beta - \gamma) \quad (10)$$

In  $(\Delta \rightarrow p\gamma)_0$  we omitted the  $\eta$  term of Ref.[2] for the reasons explained in the first of the two Refs.[17]. The fact that experimentally  $p/n$  deviates by 3% from  $-(3/2)$ , leads, using the above equations, to:  $\delta \cong (\beta + 4\gamma)$ ; neglecting for simplicity the 3% deviation, we set  $\delta = (\beta + 4\gamma)$  and get for  $(\Delta \rightarrow p\gamma)_0$  (10):

$$(\Delta \rightarrow p\gamma)_0 = (2/3)\sqrt{2}p(1 + 3[(\beta - \gamma)/p]) \quad (11)$$

The evaluation [2,18-20] of the effect of the transferred momentum ( $k=260$  MeV) depends on some assumptions, but points to  $3[(\beta - \gamma)/p]$  in (11)  $\approx 0.45$ . Even if  $\beta$  and  $\gamma$  are small (say  $\approx 0.2$ ), but have opposite signs (with  $\beta > 0$ ), one can get  $3[(\beta - \gamma)/p] \approx 0.45$ ; thus  $(\Delta \rightarrow p\gamma)_0 \approx 1.45(2/3)\sqrt{2}p$ . We underline: The factor  $3(\beta - \gamma)/p$  in Eq.(11) contains only quantities ( $\beta$  and  $\gamma$ ) of second order in

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<sup>3</sup>In [2] correct the following misprints:In Eq.(63) insert  $(-2\gamma)$  in the second square brackets; in Eq.(66) write  $F=\delta - \beta - 4\gamma$ ; in Eq.(64)(Q term) replace  $-2\gamma$  with  $+4\gamma$ .



the hierarchy but their opposite signs and the factor 3 produce a factor 0.45. To summarize, the values of  $p, n$  and  $(\Delta \rightarrow p\gamma)_0$  plus that of  $\delta$  (8) determine  $\alpha, \beta, \delta, \gamma$ ; a determination near to that below (Eq.(12)) produces a good fit and indicates a hierarchy:

$$\alpha = 3.05, \quad \delta = -0.61, \quad \beta = 0.21, \quad \gamma = -0.185 \quad (12)$$

This description *confirms the idea* (see ref.6f in [3];also [17]) *that the perfection of the ratio*  $(p/n) = -3/2$ , so important [5] for the acceptance in 1965 of the quark description,*is due to an accident*. Indeed, because the main correction to  $p/n = -3/2$  comes from the two index term  $\delta$ , one would expect, a priori, that the experimental deviation of  $(p/n)$  from  $-(3/2)$  should be 20%, not 3%. Then to get  $(\delta - \beta - 4\gamma) \cong -0.08$ , instead of  $\approx 0.6$ , a cancellation of  $\delta$  (by  $-\beta - 4\gamma$ ) must take place. Again the 4 parameters of the QCD description (7) are needed to explain this; the second order parameters are important.

Note finally that an estimate of the magnetic moment of  $\Delta^+$  from (9) leads to a value from 1.5 to 2.

We can now compare the 4 parameters general parametrization just discussed with the large  $N_c$  description of the magnetic moments, which contains 2 or 3 parameters [14-16]. The main point is this: The neglect in [14-16] of terms of order  $1/N_c^2$ , clearly corresponds to the omission of  $\beta$  and  $\gamma$  in our Eqs.(9) to (11). The previous analysis shows that  $\beta$  and  $\gamma$  are necessary.

## 5. Conclusion.

The contents of Sects.3 and 4 illustrate two reasons of unreliability of the large  $N_c$  description for  $N_c = 3$ . 1) A given term in the  $1/N_c$  treatment may be a combination of several contributions present in 3-color QCD; 2) We showed, for the magnetic moments, that terms of second order in the hierarchy are *essential* to account for the facts.

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